IV. Fourier Transform (FT)

Extension of Fourier series concept to FT
Dirichlet's conditions for FT
Periodic signal FT
FT properties
- Linearity
- Symmetry
- Time-shifting
- Differentiation/integration
- Frequency & time scaling
- Duality

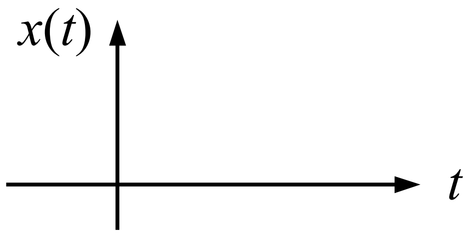
Examples
Parseval's relation
Convolution property
Partial fraction expansion
Modulation property
Applications to Communications
Applications to the AM superheterodyne receiver
Applications to frequency division multiplexing

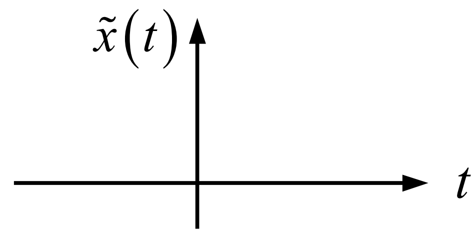
IV. Fourier Transform

Basic Idea

- represent a non-periodic signal
- use this representation to describe the effect of LTI systems

1) Extension of the Fourier Series Concept:





2) Conditions for Existence of Fourier Transform ==> "Dirichlet's Conditions"

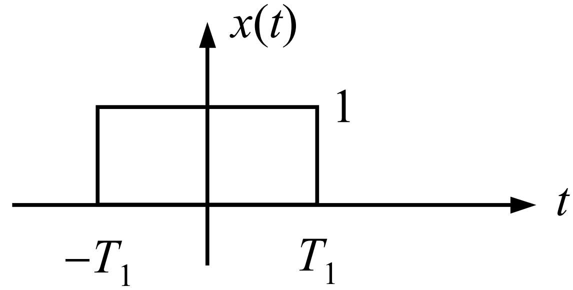
(1) $x(t)$ is absolutely integrable

(2) $x(t)$ has a finite number of maxima and minima within any finite interval

(3) $x(t)$ as a finite number of discontinuities within a finite interval

Exceptions:

Example: $x(t) = te^{-at}u(t)$

$x(t) =$ 

3) Important Fourier Property:

===> 1-to-1 Correspondence

Examples:

$$x(t) = \delta(t)$$
$$x(t) = 1$$
$$x(t) = \text{sgn}\,(t)$$
$$x(t) = u(t)$$

4) Fourier Transform of Periodic Signal

<u>Note</u>:

- periodic signal doesn't have a Fourier transform (following Dirichlet's conditions)
- one considers it has a spectrum/line spectrum
- derivation done by analogy with Fourier series decomposition

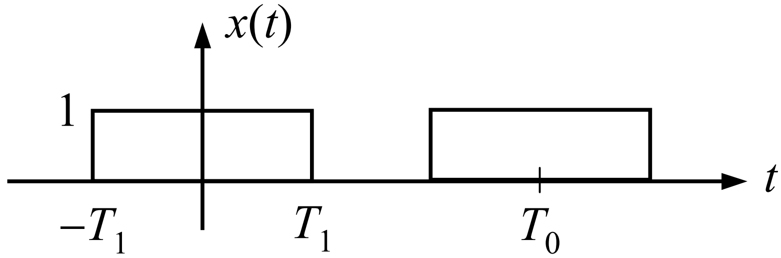
Recall:

$$F_T\left\{1\right\} = 2\pi\delta\left(\omega\right)$$

$$F_T\left\{1 \cdot e^{j\omega_0 t}\right\} =$$

$$x\left(t\right) = \sum a_k e^{jk\omega_0 t}$$

Example:



$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT) \quad ; \quad y(t) = \cos(\omega_0 t)$$

$$z(t) = \cos(\omega_0 t + \pi / 4)$$

5) Fourier Transform Properties

(a) Linearity

$$a x_1(t) + b x_2(t) \xrightarrow{F_T}$$

(b) Symmetry

- if x(t) is real:
- proof:
- consequences:

- properties:

$$x(t) \text{ real + even } \Rightarrow X(\omega) \text{ real + even}$$
$$x(t) \text{ real + odd } \Rightarrow X(\omega) \text{ imaginary + odd}$$

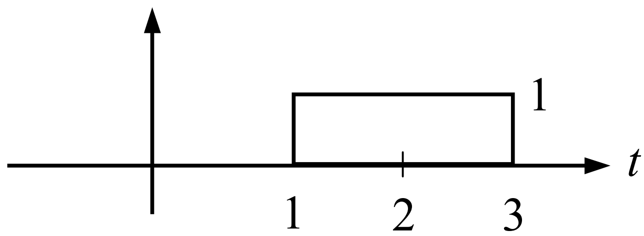
(c) <u>Time Shifting</u> $x(t) \rightarrow \mathrm{x}(\omega)$

$x(t - t_0) \rightarrow$

- proof

- consequence:

Example:

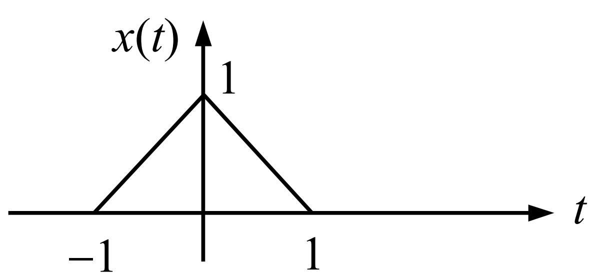


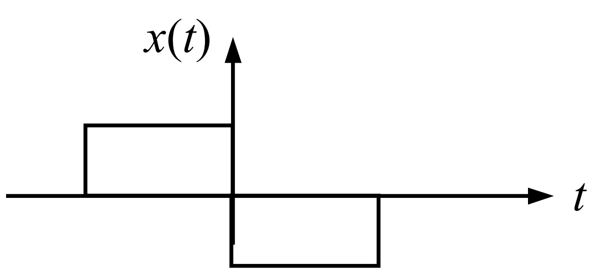
(d) <u>Differentiation/Integration</u>

$$x(t) = \frac{1}{2\pi} \int X(\omega) e^{j\omega t} d\omega$$

$$\frac{d}{dt} x(t)$$

<u>Example</u>:





(d) Integration

$$\int_{-\infty}^{t} x(\tau)\, d\tau \rightarrow$$

Example:

$$\int_{-\infty}^{t} \delta(t)dt$$

(e) Frequency and Time Scaling (similarity theorem)

- $x(at) \rightarrow$

- proof:

Example: $f(t) = \text{sinc}^2(t)$

(f) <u>Duality</u>

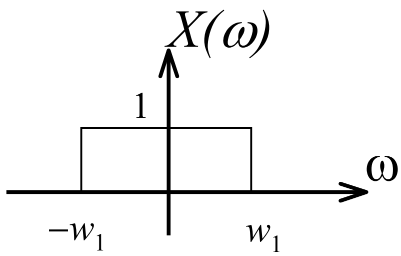
Remember $$x(t) = \frac{1}{2\pi}\int X(\omega)e^{j\omega t}d\omega$$

$$X(\omega) = \int x(t)e^{-j\omega t}dt$$

Property: *x(t)* --> *X(ω)*
X(t) --> *2πx(-ω)*

• Proof:

Example:



$X(t)$?

Examples: Compute the FT or IFT using duality properties

$$X(\omega) = \cos(5\omega)$$

$$Y(\omega) = \cos(5\omega + \pi / 4)$$

$$z(t) = e^{-jt}$$

$$v(t) = \frac{1}{1 + jt}$$

$$c(t) = \delta(t)$$

Basic Fourier Transform Properties

$$\delta(t) \;\rightarrow\; 1$$

$$s \;\rightarrow\; 2\pi\delta(\omega)$$

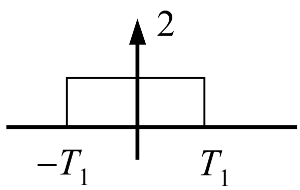
$$\text{sgn(t)} \;\rightarrow\; \frac{2}{j\omega}$$

$$u(t) \;\rightarrow\; \pi\delta(\omega) + j\omega$$

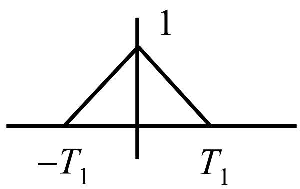
$$e^{j\omega_0 t} \;\rightarrow\; 2\pi\delta(\omega - \omega_0)$$

$$\cos(\omega_0 t) \;\rightarrow\; \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$$\sin(\omega_0 t) \;\rightarrow\; j[\pi\delta(\omega + \omega_0) - \pi\delta(\omega + \omega_0)]$$



$$\rightarrow\; 2T_1\,\text{sinc}\left(\frac{\omega T_1}{\pi}\right) = \frac{2\sin\omega T_1}{\omega}$$



$$\rightarrow\; T_1\left[\,\text{sinc}\left(\frac{\omega T_1}{2\pi}\right)\right]^2$$

$$ax_1(t)+bx_2(t) \;\rightarrow\; a_1X_1(\omega)+b_1X_2(\omega)$$

$$x(t-t_0) \;\rightarrow\; X(\omega)e^{-j\omega t_0}$$

$$\frac{d}{dt}x(t) \;\rightarrow\; j\omega X(\omega)$$

$$\frac{d^n x(t)}{dt^n} \;\rightarrow\; (j\omega)^n X(\omega)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \;\rightarrow\; \frac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega)$$

$$x(at) \;\rightarrow\; \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$$

$$X(t) \;\rightarrow\; 2\pi x(-\omega)$$

$$x(t)e^{j\omega_0 t} \rightarrow X(\omega-\omega_0)$$

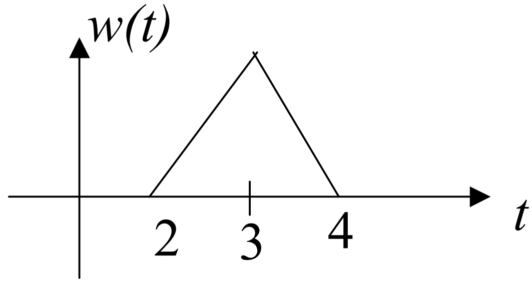
Example:

x(t)=cos(3t) -2<t<2, =0 ow

y(t)=2/(5+2jt)

z(t)=2/πt

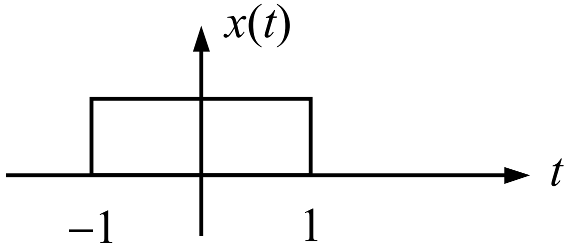
v(t)=2δ(t+3)-4δ(t-3)



6) Parseval's Relation (conservation of energy)

- <u>Property</u>: $\int_{-\infty}^{+\infty} \left| x(t) \right|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| X(\omega) \right|^2 d\omega$
- <u>What does this mean</u>?
- <u>Proof</u>:
- <u>What does this mean for periodic signals</u>?

- Example

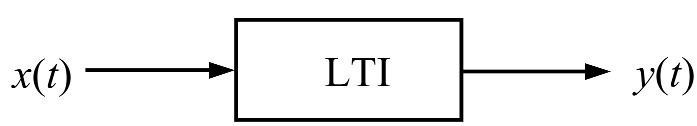


7) Convolution Property

$$x(t) * g(t) \quad \rightarrow \quad F(\omega) \cdot G(\omega)$$

- Proof:

- <u>Application to LTI systems</u>



- Example

$$x(t) = e^{-bt}u(t) \qquad\qquad y(t) = ?$$

$$h(t) = e^{-at}u(t)$$

- Example

$$x(t) = te^{-3t}u(t)$$

$$h(t) = e^{-t}u(t)$$

Partial Fraction Expansion

$$H(\omega)=\frac{A(\omega)}{(j\omega+a)^p}=\frac{\alpha_p}{(j\omega+a)^p}+\frac{\alpha_{p-1}}{(j\omega+a)^{p-1}}+\cdots+\frac{\alpha_1}{j\omega+a}$$

where:

$$\alpha_p=H(\omega)(a+j\omega)^p\Big|_{j\omega=-a}$$

$$\alpha_{p-1}=\frac{1}{j}\frac{d}{d\omega}\Big[H(\omega)(a+j\omega)^p\Big]\Big|_{j\omega=-a}$$

$$\vdots$$

$$\alpha_1=\frac{1}{(p-1)!}\frac{1}{j^{p-1}}\frac{d^{p-1}}{d\omega^{p-1}}\Big[H(\omega)(a+j\omega)^p\Big]\Big|_{j\omega=-a}$$

Example:

$$H(\omega)=\frac{j\omega}{(1+j\omega)^2(2+j\omega)}=$$

8) Modulation Property

$$y(t) = x(t).h(t) \;\; \rightarrow$$

- Proof:

- <u>Application of modulation property</u>

$$p(t) = s(t) \cdot \cos \omega_0 t$$

$$\swarrow$$

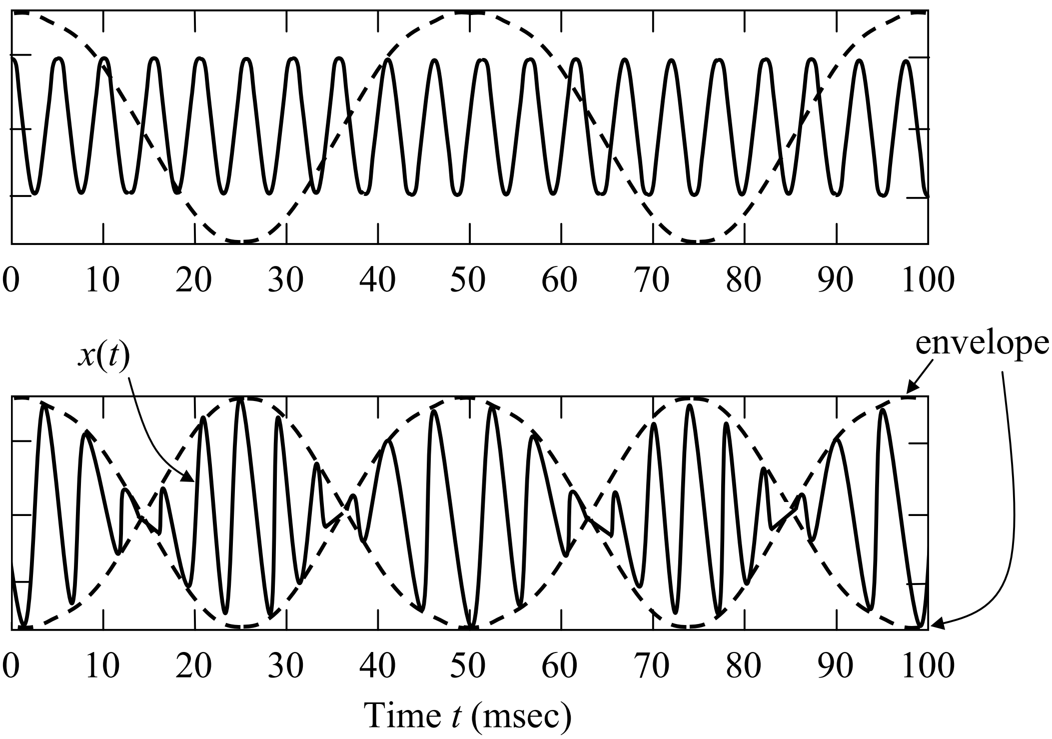
$$P(\omega) =$$

- Applications: communication systems; Amplitude modulation (AM) systems.
- Speech exist in the range 300Hz~5KHz
- Atmosphere attenuates signals rapidly in the range 10Hz-->20KHz, and propagates much better at high frequencies

$\Rightarrow$ shift speech to higher frequency range

• What does the AM signal look like ?

Example: $x(t)=\cos(40\pi t)\cdot\cos(400\pi t)$



$$x(t)=\frac{1}{2}\left[\cos(440\pi t)+\cos(360\pi t)\right]$$

$$=\frac{1}{4}\left[e^{j440\pi t}+e^{-j440\pi t}+e^{-j360\pi t}+e^{-j360\pi t}\right]$$

Overall period of signal $x(t)$

•

•

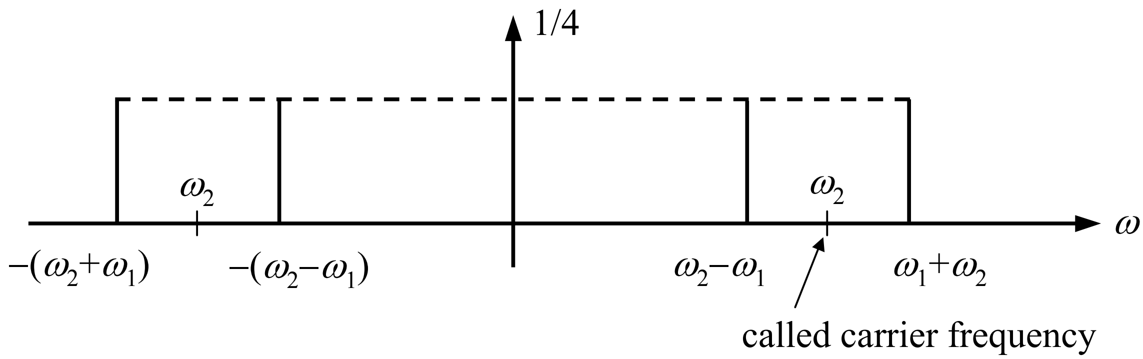
overall period
$T=$

$$\Rightarrow \quad x(t) = \sum_k a_k e^{jk\omega_0 t}$$

<u>Spectrum of $x(t)$</u>:

Spectrum of $x(t)$ for generic frequencies:

$$y(t) = \cos(\omega_1 t)\cos(\omega_2 t) \qquad \omega_2 \gg \omega_1$$
$$= \frac{1}{2}\left[\cos(\omega_1+\omega_2)t + \cos(\omega_1-\omega_2)t\right]$$
$$= \frac{1}{4}\left\{\exp\left[j(\omega_1+\omega_2)t\right] + \exp\left[-j(\omega_1+\omega_2)t\right]\right.$$
$$\left. + \exp\left[j(\omega_1-\omega_2)t\right] + \exp\left[-j(\omega_1-\omega_2)t\right]\right\}$$



Note:

Change $\omega_2 \rightarrow$ you change where the frequency's components are for a constant ω_1

- How to recover the original speech signal ?

  Demodulate….

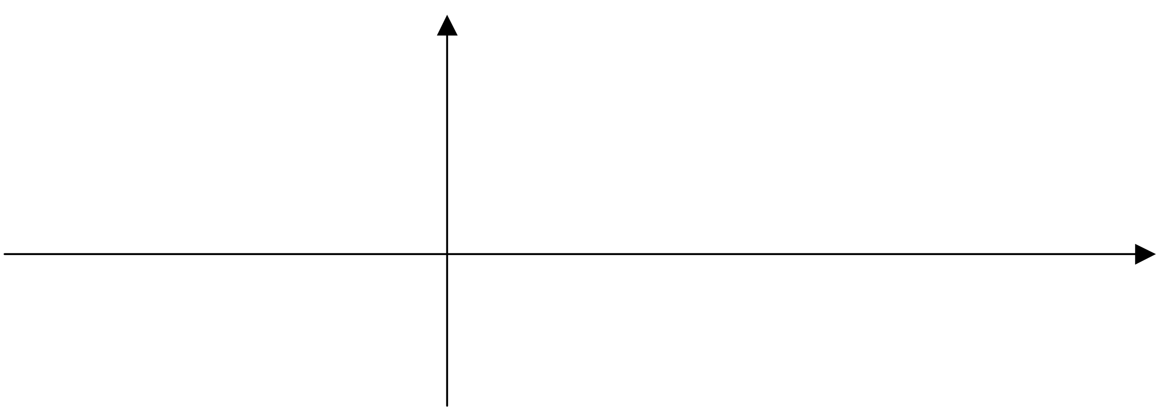
9) Applications to the AM superheterodyne receiver

Several basic operations are needed in a broadcast receiver:

1. Station separation: must be able to pick out a specific signal and reject others
2. Amplification: needed when the signal picked up by the radio antenna is too weak to drive the loudspeakers
3. Demodulation: The received signal is centered around the carrier frequency and must be demodulated before it is fed into the speakers

In standard AM:

- The maximum audio signal frequency is around 5kHz
- Each station is assigned 10kHz by the FCC (i.e., each adjacent carriers are separated by 10kHz)
- The AM frequency band assignment is 540kHz --> 1600kHz



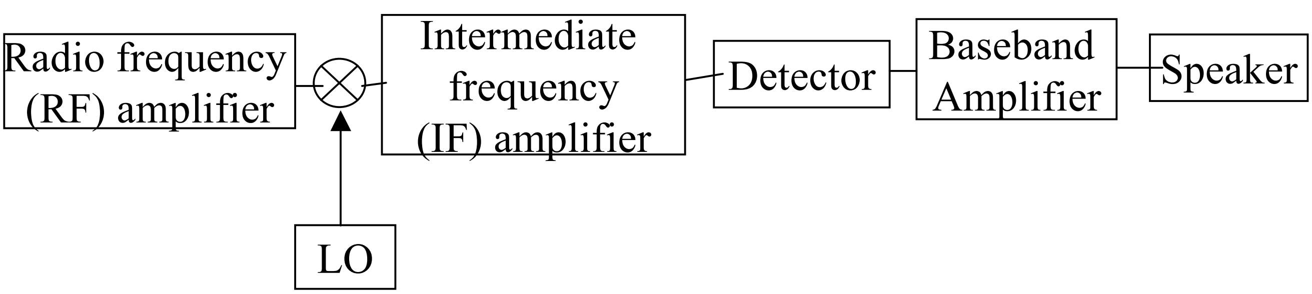
• **Filter constraints:** We need tuneable filters with sharp cutoff frequencies to select the station we want

impossible to realize!

• **What is done instead:** We build a fixed bandpass filter and shift the input frequencies so that the frequencies of interest falls within the fixed passband of the filter

- Such a shifting process is called *heterodyning*
- The receiver doing this operation is called a *superheterodyne receiver*

• **Basic AM superheterodyne receiver diagram**



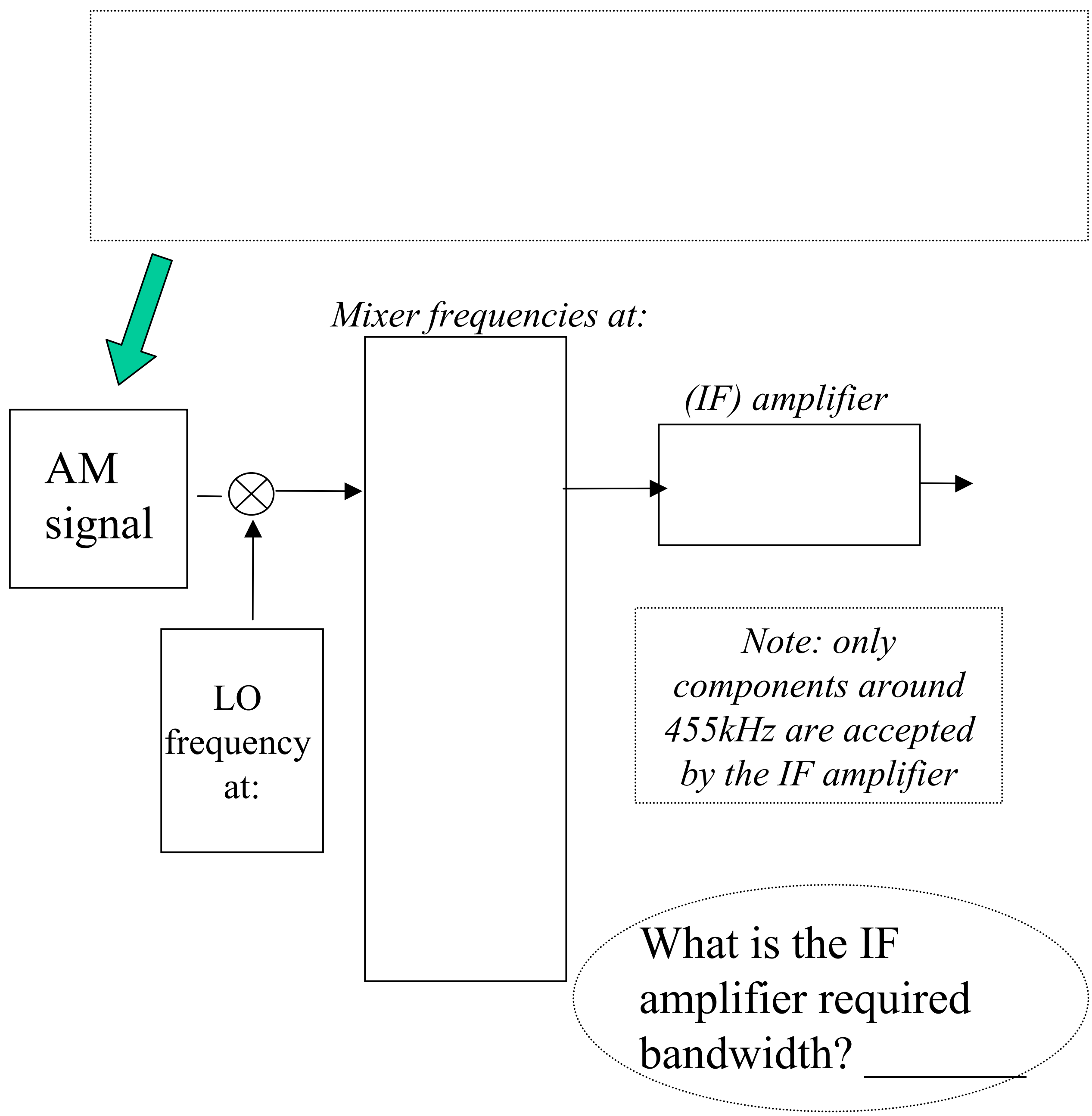
Receiver Components:

- RF amplifier: amplifies a portion of the spectrum (tuneable)
- Local oscillator (Mixer): shifts the signal to a specific frequency range
- IF amplifier: filters and amplifies around a fixed freqency (for AM systems around 455kHz)
- Detector: demodulates (i.e., extracts) the audio signal
- Baseband amplifier: amplifies the audio signal

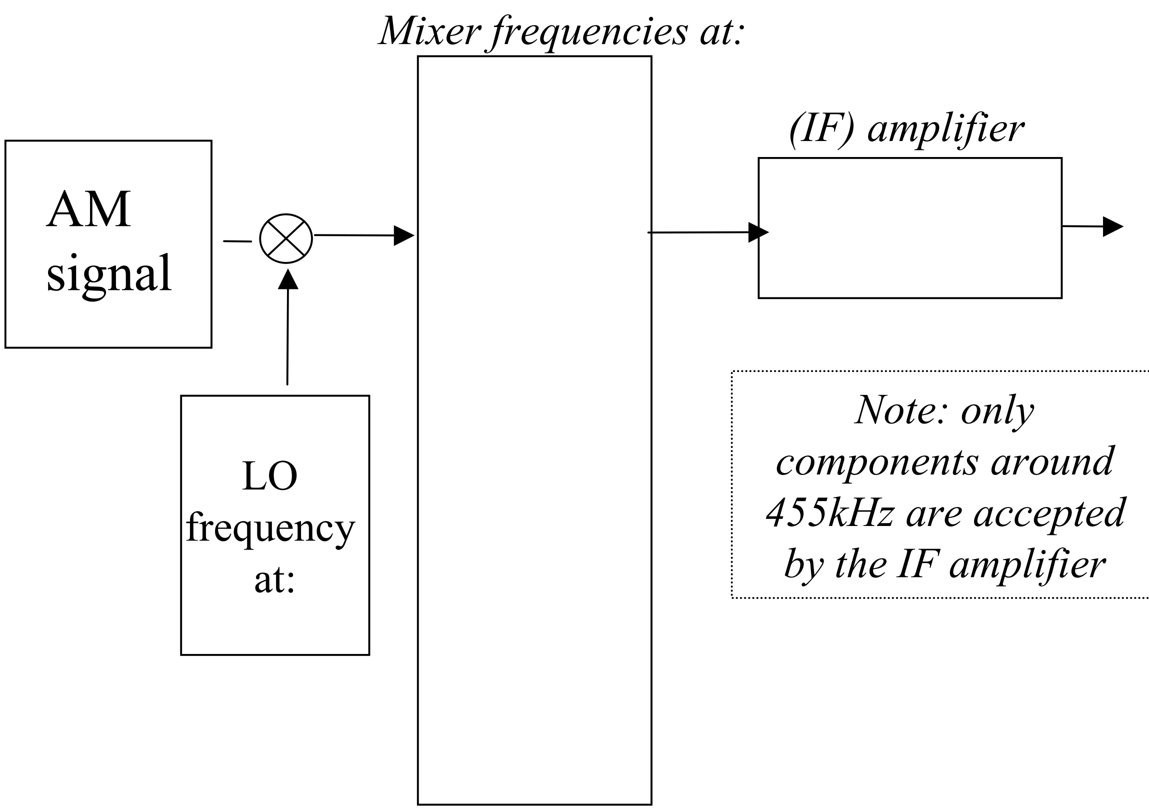
Example:

- Assume no RF amplifier (will be added and discussed later)
- Consider the case of a 1kHz AM wave modulated by a carrier at 1MHz (i.e., the station center frequency is at 1MHz)

The generated AM signal has frequencies at:



• What happens if we want to accept a station located at 1600kHz ? Assume the IF amplifier is centered around 455KHz.



Summary:

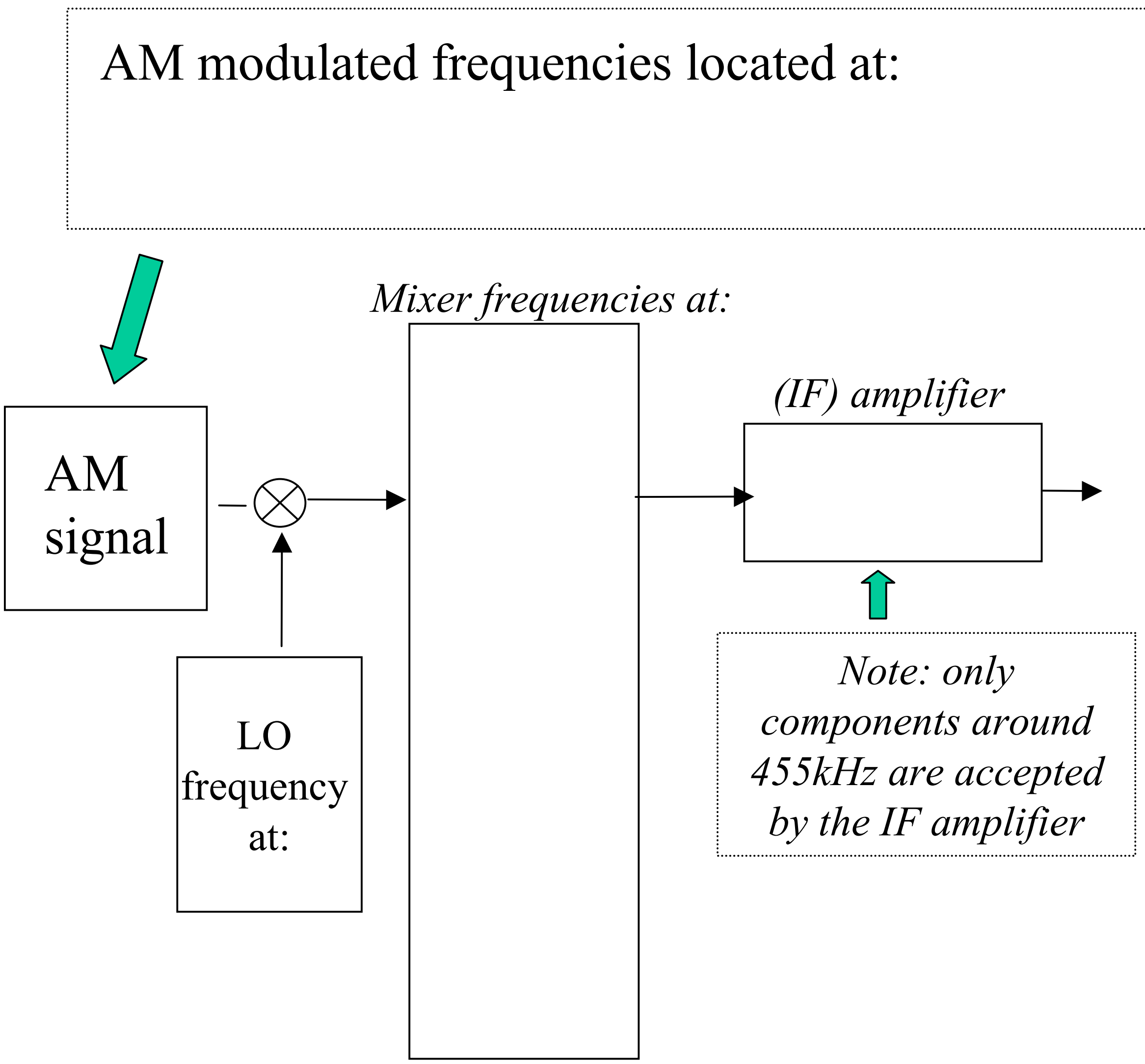
- The key is to make the LO track the incoming signal so hat the difference between the incoming signal and the LO frequency is a constant frequency (called the IF frequency) equal to 455KHz.

- By convention the LO has to be at a frequency 455KHz above the incoming carrier frequency.

- Once we have the IF amplifier output, we can demodulate.

•**The Potential *Image Frequency* problem:** Sometimes we can get a signal other than that desired at the IF amplifier

- Example: Assume we have a desired signal of frequency 1KHz modulated at 620KHz and an undesired signal of frequency 1 KHz modulated at 1530KHz

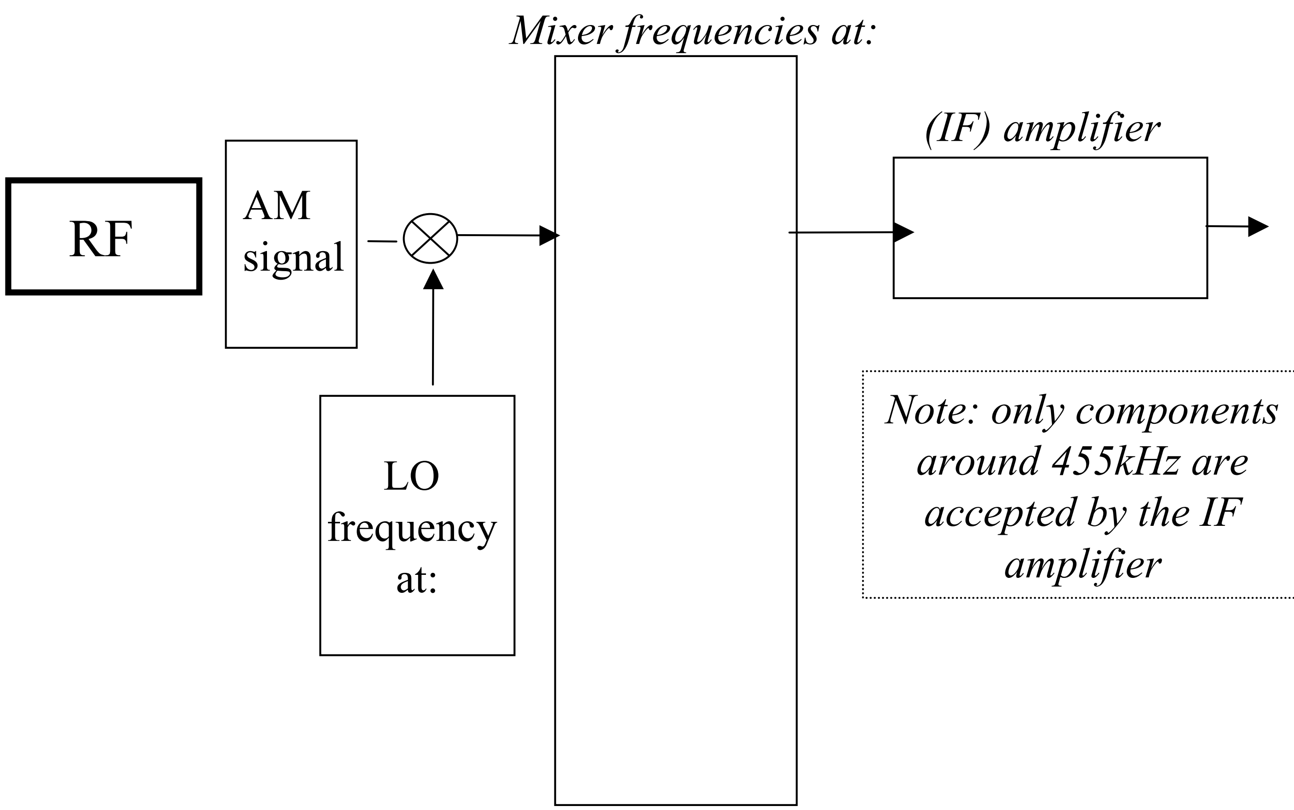


• **Notes**: 1. Both desired and undesired modulated signals have identical components after the mixer. These components won't be separated.

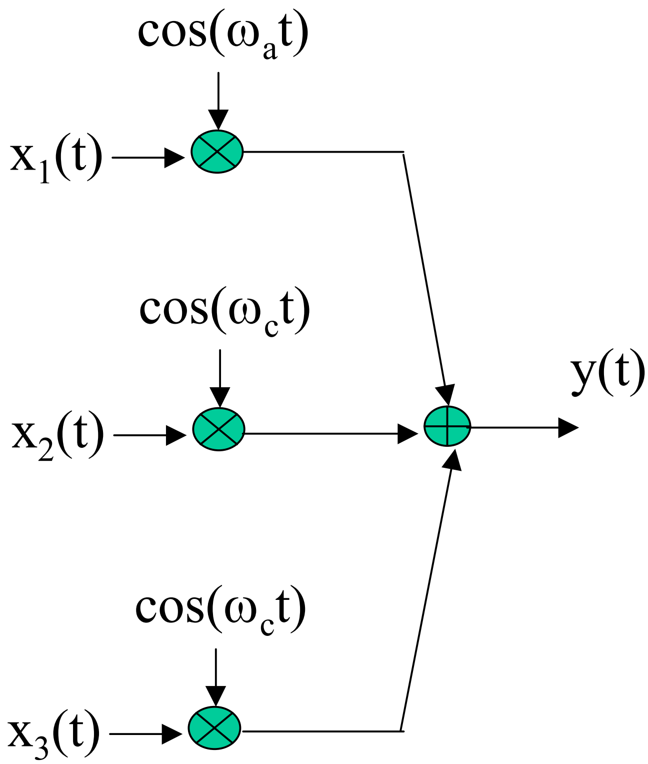
2. Such undesired modulated components are called *image frequencies* (they are the frequencies which appear in the correct range to the IF amplifier, while they are undesirable to start with.

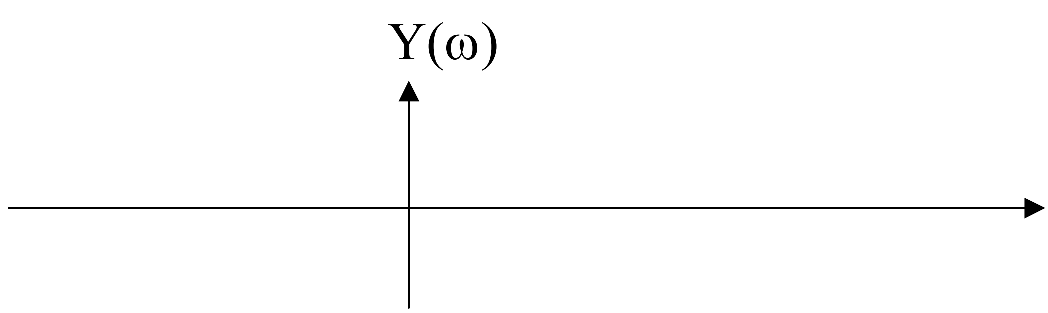
• **Question**: how to determine the image frequency which will be a problem to a specific AM signal ?

• How to use the RF amplifier to remove the image frequency problem



- **Application to Frequency Division Multiplexing**





- Demodulation